Theorems and Applications for Roots of Polynomials

Rational Root Theorem: Given $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, where each coefficient is an integer. Two different types of roots possible.

- All of the integer roots must be factors of a_0.
- All rational roots (fractions) must have reduced form $\frac{p}{q}$ where *p* is integer factor of a_0 and *q* is integer factor of a_n.
- Note: Once you determine one root, you can use long division or synthetic division to reduce polynomial to quadratic level, and then you can use quadratic formula, standard factoring, etc. to factor completely.

Examples

- What are the rational roots of $x^3 - 5x^2 - 2x + 10 = 0$?

- What are the rational roots of $x^3 - 2x^2 - 5x + 6 = 0$? Rewrite polynomial in factored format.

Fundamental Theorem of Algebra: If $P(x)$ is polynomial of degree $n \geq 1$, then $P(x) = 0$ has exactly *n* roots/solutions/zeros, including multiple and complex roots.

Examples

- What are the complex roots of $x^5 - x^4 - 7x^3 + 7x^2 - 18x + 18 = 0$?

- What are the zeros of $f(x) = x^4 + 2x^3 - 4x^2 - 7x - 2$?

Complex Zero's: Occur in Conjugate Pairs. If $a + bi$ is a zero of a function, then $a - bi$ is also a zero of the function.

EX: Find the quartic polynomial function with real coefficients that has 1, -2, and $2i$ as zeros, and $f(-1) = 10$.

Binomial Theorem and Pascal's Triangle: Used to expand powers of binomials.

$$
\begin{aligned}
(a+b)^0 &= 1\\
(a+b)^1 &= a+b\\
(a+b)^2 &= a^2+2ab+b^2\\
(a+b)^3 &= a^3+3a^2b+3ab^2+b^3\\
(a+b)^4 &= a^4+4a^3b+6a^2b^2+4ab^3+b^4\\
(a+b)^5 &= a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5
\end{aligned}
$$

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 \\
\therefore & & & & \vdots & & & & & & & & \ddots
\end{array}
$$

- Binomial Theorem: $(a+b)^n = P_0a^n + P_1a^{n-1}b + P_2a^{n-2}b^2 + \cdots + P_{n-1}ab^{n-1} + P_nb^n$

Examples

- What is the expansion of $(x+3)^4$?

- What is the expansion of $(3-z)^3$?

- What is the linear model that best fits the data below? Use the model to estimate the egg consumption in 1995.

Year	**US Per Capita Egg Consumption**
1970	302
1980	266
1990	231
2000	247

- What is the quadratic model that best fits the data above? Use the model to estimate the egg consumption in 1995.

- Which is the better model for estimating egg consumption between 1970 and 2000?